Lecture 17 3/25/2019

Electromagnetic Plane Waves (Cont'd)

Total Internal Reflection

In our discussion so far, we have assumed that $n'^2 > n^2 \sin^2\theta$. This is always the case if $n' > n$. However, if $n' < n$, then $n'^2 - n^2 \sin^2\theta < 0$ for $\theta > \theta_C \equiv \sin^{-1}\left(\frac{n'}{n}\right)$. As a result, the refraction angle is not real in this case. The refracted wave then has the electric field of the following form:

$$\vec{E}' = \vec{E}'_0 \, e^{i k'_\perp z} \, e^{i(\vec{k}'_\parallel \cdot \vec{x} - \omega t)}$$

Where $k'_\perp = \frac{i\omega}{c}\sqrt{n^2 \sin^2\theta - n'^2}$ and $k'_\parallel = k_\parallel = \frac{\omega}{c} n \sin\theta$. Therefore, $k'_\perp$ is purely imaginary, and we have:

$$\vec{E}' = \vec{E}'_0 \, e^{-|k'_\perp| z} \, e^{i(\vec{k}'_\parallel \cdot \vec{x} - \omega t)}$$

This describes an inhomogeneous plane wave whose amplitude decays in a direction that is perpendicular to the interface, while its front moves along the interface. We note that:

$\vec{\nabla}\cdot\vec{E}'=0 \Rightarrow \vec{k}'\cdot\vec{E}_0'=0 \Rightarrow \vec{k}_{\parallel}'\cdot\vec{E}_{0\parallel}' + i|k_z'|E_{0z}'=0$

Hence, E_{0z}' and $\vec{k}_{\parallel}'\cdot\vec{E}_{0\parallel}'$ have a $\frac{\pi}{2}$ phase difference. The energy flow normal to the interface is given by (after time-averaging):

$$S'=\frac{1}{2}Re(\vec{E}'\times\vec{H}'^{*})\cdot\hat{z}=\frac{1}{2}Re\left[\vec{E}'\times\frac{(\vec{k}'^{*}\times\vec{E}'^{*})}{\omega\mu'}\right]\cdot\hat{z}=\frac{1}{2\omega\mu'}Re\left[(\vec{E}'\cdot\vec{E}'^{*})(\vec{k}'^{*}\cdot\hat{z})-(\vec{k}'^{*}\cdot\vec{E}')(\vec{E}'^{*}\cdot\hat{z})\right]$$

The first term inside the bracket is purely imaginary, thus:

$$S'=-\frac{1}{2\omega\mu'}Re\left[(\vec{k}'^{*}\cdot\vec{E}')E_z'^{*}\right]$$

For "s polarization", $E_z'=0$ implying that $S'=0$. For "p polarization", however, $E_z'\neq 0$. In this case, we can use:

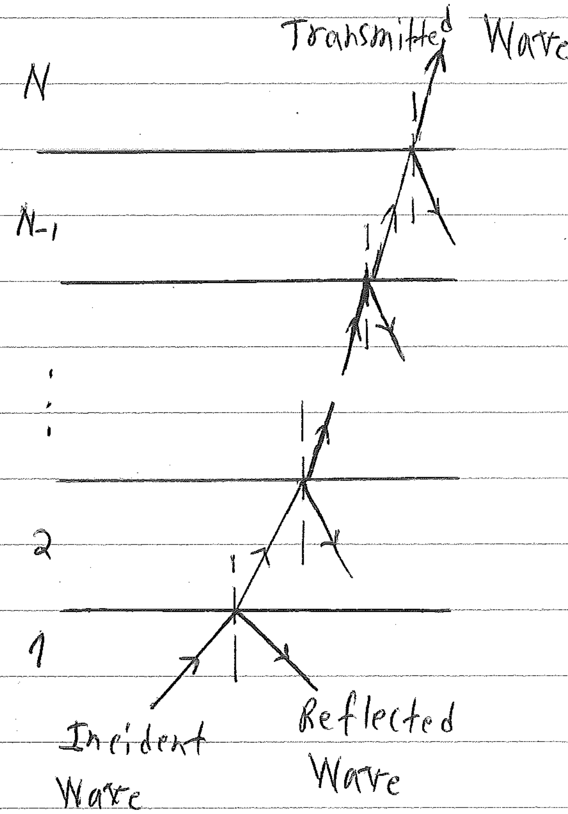
$$\underbrace{\vec{k}'\cdot\vec{E}'=0}_{\vec{\nabla}\cdot\vec{E}'=0} \Rightarrow k_z'E_z'+\vec{k}_{\parallel}'\cdot\vec{E}_{\parallel}'=0 \Rightarrow \vec{k}_{\parallel}'\cdot\vec{E}_{\parallel}' \underset{\vec{k}_{\parallel}' \text{ is real}}{=} \vec{k}_{\parallel}'^{*}\cdot\vec{E}_{\parallel}'=-k_z'E_z'$$

This leads to:

$$S'=-\frac{1}{2\omega\mu'}Re\left[(\vec{k}'^{*}\cdot\vec{E}')E_z'^{*}\right]=-\frac{1}{2\omega\mu'}Re\left[-2i|k_z'|E_z'^{2}\right]=0$$

Therefore, as expected, there is no energy flow normal to the interface when we have total internal reflection.

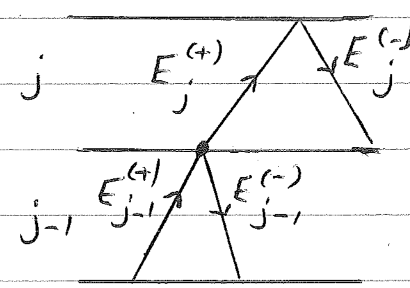
Multiple Parallel Interfaces



Considering N media, we have N-2 internal layers and N-1 interfaces. In each layer, and the first medium, two plane wave exist. While, in the final medium there is only the transmitted wave.

Let us consider the interface that separates layers j-1 and j:

We have:



$$E_j^{(+)} = t_{j-1,j}\, E_{j-1}^{(+)} + r_{j,j-1}\, E_j^{(-)}$$

$$E_{j-1}^{(-)} = r_{j-1,j}\, E_{j-1}^{(+)} + t_{j,j-1}\, E_j^{(-)}$$

Here, t_{nm} is the amplitude transmission coefficient from layer "n" to layer "m", and r_{nm} is the amplitude reflection coefficient from layer "n" to layer "m". The quantities $E_j^{(+)}$, $E_j^{(-)}$, $E_{j-1}^{(+)}$, $E_{j-1}^{(-)}$ are the wave values at the interface between the layers j and $j-1$. After using the relations $r_{j,j-1} = -r_{j-1,j}$ and $r_{j,j-1}^2 + t_{j,j-1}\, t_{j-1,j} = 1$, we can write $E_j^{(\pm)}$ in terms of $E_{j-1}^{(\pm)}$ as follows;

$$\begin{bmatrix} E_j^{(+)} \\ E_j^{(-)} \end{bmatrix} = \begin{bmatrix} \dfrac{1}{t_{j,j-1}} & \dfrac{r_{j,j-1}}{t_{j,j-1}} \\ \dfrac{r_{j,j-1}}{t_{j,j-1}} & \dfrac{1}{t_{j,j-1}} \end{bmatrix} \begin{bmatrix} E_{j-1}^{(+)} \\ E_{j-1}^{(-)} \end{bmatrix}$$

The 2×2 matrix, denoted by $\mathbb{T}^{(j)}$ is called the transfer matrix of the j-th interface.

We also need to take propagation in layer j into account. This is done by considering the phase difference at the top and bottom of that layer;

$$\begin{bmatrix} E_j^{(+)}(\text{top}) \\ E_j^{(-)}(\text{top}) \end{bmatrix} = \underbrace{\begin{bmatrix} e^{i k_z^{(j)} d_j} & 0 \\ 0 & e^{-i k_z^{(j)} d_j} \end{bmatrix}}_{\mathbb{P}^{(j)}} \begin{bmatrix} E_j^{(+)}(\text{bottom}) \\ E_j^{(-)}(\text{bottom}) \end{bmatrix}$$

Here, d_j is the thickness of the j-th layer.

We can then write:

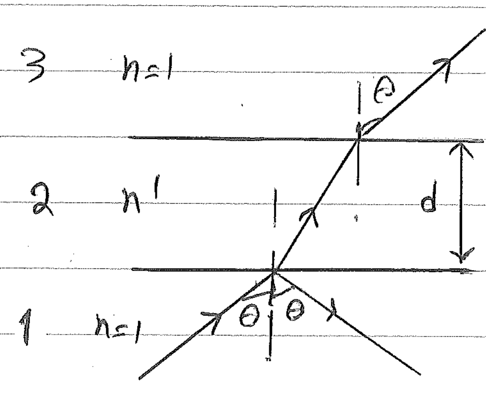
$$\begin{bmatrix} E_N^{(+)} \\ E_N^{(-)} \end{bmatrix} = \mathbb{I}^{(N-2)}\,\mathbb{P}^{(N-2)}\,\mathbb{I}^{(N-3)}\,\mathbb{P}^{(N-3)} \cdots\cdots\, \mathbb{I}^{(0)} \begin{bmatrix} E_1^{(+)} \\ E_1^{(-)} \end{bmatrix}$$

Here $E_N^{(+)}$ is the transmitted wave, $E_N^{(-)}=0$, $E_1^{(-)}$ is the reflected wave, and $E_1^{(+)}$ is the incident wave. This leaves us with two equations for two unknowns ($E_1^{(-)}$ and $E_N^{(+)}$).

We note that in above we have assumed "s polarization" or "p polarization" with the corresponding r_{nm} and t_{nm} coefficients. A general polarization can be written as a superposition of these polarizations and treated accordingly.

Example: A simple dielectric layer, polarization perpendicular to the plane of incidence.

In this case, we have:



$$r_{12} = \frac{n\cos\theta - n'\cos\theta'}{n\cos\theta + n'\cos\theta'} = -r_{21}$$

$$t_{12} = \frac{2n\cos\theta}{n\cos\theta + n'\cos\theta'} \quad , \quad t_{21} = \frac{2n'\cos\theta'}{n\cos\theta + n'\cos\theta'}$$

Also:

$$r_{23} = r_{21} \;,\; t_{23} = t_{32} \;,\; r_{32} = r_{12} \;,\; t_{32} = t_{12}$$

Then:

$$\mathbb{I}^{(0)} = \begin{bmatrix} \frac{1}{t_{21}} & \frac{r_{21}}{t_{21}} \\ \frac{r_{21}}{t_{21}} & \frac{1}{t_{21}} \end{bmatrix} , \quad \mathbb{I}^{(1)} = \begin{bmatrix} \frac{1}{t_{12}} & \frac{r_{12}}{t_{12}} \\ \frac{r_{12}}{t_{12}} & \frac{1}{t_{12}} \end{bmatrix} , \quad \mathbb{P}^{(1)} = \begin{bmatrix} e^{in'k\cos\theta' d} & 0 \\ 0 & e^{-in'k\cos\theta' d} \end{bmatrix}$$

Thus:

$$\begin{bmatrix} E_3^{(+)} \\ 0 \end{bmatrix} = \underbrace{\mathbb{I}^{(1)}\,\mathbb{P}^{(1)}\,\mathbb{I}^{(0)}}_{\mathbb{I}} \begin{bmatrix} E_1^{(+)} \\ r E_1^{(+)} \end{bmatrix}$$

$$\mathbb{I} = \begin{bmatrix} \frac{e^{i\alpha}}{t_{12}t_{21}} + e^{-i\alpha}\frac{r_{12}r_{21}}{t_{12}t_{21}} & e^{i\alpha}\frac{r_{21}}{t_{12}t_{21}} + e^{-i\alpha}\frac{r_{12}}{t_{12}t_{21}} \\ e^{i\alpha}\frac{r_{12}}{t_{12}t_{21}} + e^{-i\alpha}\frac{r_{12}}{t_{12}t_{21}} & e^{i\alpha}\frac{r_{12}r_{21}}{t_{12}t_{21}} + \frac{e^{-i\alpha}}{t_{12}t_{21}} \end{bmatrix}$$

Where: $(\alpha \equiv n'k\cos\theta' d)$

$$\boxed{r = \frac{2ir_{12}\sin\alpha}{(e^{-i\alpha} - e^{i\alpha}r_{12}^2)}}$$

It is seen that $r=0$ if $\alpha = m\pi$ (m an integer), hence no reflection. In this case the incident wave is completely transmitted.

The condition for this to happen is:

$$2dn'\cos\theta' = m\lambda \Rightarrow d\cos\theta' = \frac{m\lambda}{2n'}$$

This phenomenon, called "transmission" or "Fabry-Perot resonance", can be understood as a result of destructive interference between the directly reflected wave at the 1-2 interface and the wave refracted at that interface then reflected at the 2-3 interface and then refracted back to 1.